A SIMPLE PROOF OF HEAVY BALL CONVERGENCE

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Let $f(w)$ be a scalar function of a point w in euclidean space. A basic problem is to minimize $f(w)$, that is, to find or compute a minimizer w^*,

$$f(w) \geq f(w^*), \qquad \text{for every } w.$$

A *descent sequence* is a sequence w_0, w_1, w_2, $\dots$ satisfying

$$f(w_0) \geq f(w_1) \geq f(w_2) \geq \dots.$$

In a descent sequence, the point after $w = w_n$ is $w^+ = w_{n+1}$, and the point before w is $w^- = w_{n-1}$. Then $(w^-)^+ = w = (w^+)^-$.

We assume the loss function is quadratic,

$$f(w) = \frac{1}{2} w \cdot Qw - b \cdot w, \tag{1}$$

where the eigenvalues of the symmetric matrix Q lie strictly between positive constants $m < L$. Then there is a unique global minimizer w^*. Let

$$C^2 = \max_\lambda \frac{(L-m)(L-m)}{(L-\lambda)(\lambda-m)}, \tag{2}$$

where the maximum is over the eigenvalues of Q.

Theorem (Polyak [1, 2, 3, 4])**.** *Suppose $f(w)$ is quadratic, let $r = m/L$, and set $E(w) = |w - w^*|$. Let C be given by (2). Then the descent sequence $w_{-1} = w_0$, w_1, w_2, $\dots$ given by*

$$w^+ = w - t\nabla f(w) + s(w - w^-) \tag{3}$$

with learning rate and the decay rate

$$t = \frac{1}{L} \cdot \frac{4}{(1+\sqrt{r})^2}, \qquad s = \left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)^2,$$

converges to w^* at the rate

$$E(w_n) \leq C \left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)^n E(w_0), \qquad n = 1, 2, \dots \tag{4}$$

Proof. Since $\nabla f(w) = Qw - b$, the sequence satisfies

$$w_{n+1} = w_n - t(Qw_n - b) + s(w_n - w_{n-1}), \qquad n = 0, 1, 2, \dots. \tag{5}$$

To initialize this recursion, we set $w_{-1} = w_0^- = w_0$. This implies $w_1 = w_0 - t(Qw_0 - b)$.

Let v be an eigenvector of Q with eigenvalue λ. To solve (5), we assume a solution of the form

$$w_n = w^* + \rho^n v, \qquad Qv = \lambda v. \tag{6}$$

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Inserting this into (5) and using $Qw^* = b$ leads to the quadratic equation

$$\rho^2 = (1 - t\lambda + s)\rho - s$$

with discriminant

$$\Delta = (1 - \lambda t + s)^2 - 4s.$$

Now $\Delta < 0$ exactly when

$$\frac{(1-\sqrt{s})^2}{\lambda} < t < \frac{(1+\sqrt{s})^2}{\lambda}. \tag{7}$$

If we assume

$$\frac{(1-\sqrt{s})^2}{m} \le t \le \frac{(1+\sqrt{s})^2}{L}, \tag{8}$$

then

$$\Delta \le -(1-s)^2 \frac{(L-\lambda)(\lambda-m)}{mL}, \tag{9}$$

for every eigenvalue λ of Q. When $\Delta < 0$, the roots are conjugate complex numbers ρ, $\bar{\rho}$, where

$$\rho = x + iy = \frac{(1-\lambda t + s) + i\sqrt{-(1-\lambda t + s)^2 + 4s}}{2}. \tag{10}$$

It follows the absolute value of ρ equals

$$|\rho| = \sqrt{x^2 + y^2} = \sqrt{s}.$$

To obtain the fastest convergence, we choose s and t to minimize $|\rho| = \sqrt{s}$, while still satisfying (8). This forces (8) to be an equality,

$$\frac{(1-\sqrt{s})^2}{m} = t = \frac{(1+\sqrt{s})^2}{L}.$$

These are two equations in two unknowns s, t. Solving, we obtain the choices for s and t made above.

Since (5) is a 2-step linear recursion, the general solution depends on two constants A, B. Let λ_1, λ_2, ... be the eigenvalues of Q and let v_1, v_2, ... be the corresponding orthonormal basis of eigenvectors in the euclidean space. Since (5) is a 2-step vector linear recursion, A and B are vectors, and the general solution depends on constants A_k, B_k corresponding to each λ_k, $k = 1, 2, \dots$.

If ρ_k, $k = 1, 2, \dots$, are the corresponding roots (10), then (6) is a solution of (5) for each of the roots $\rho = \rho_k$ and $\rho = \bar{\rho}_k$, $k = 1, 2, \dots$. Therefore the linear combination

$$w_n = w^* + \sum_k (A_k \rho_k^n + B_k \bar{\rho}_k^n)\, v_k, \qquad n = 0, 1, 2, \dots \tag{11}$$

is the general solution of (5). Inserting $n = 0$ and $n = 1$ into (11), then taking the dot product of the result with v_k, we obtain two linear equations for two unknowns A_k, B_k. Solving for A_k, $B_k = \bar{A}_k$, then using (9),

$$|A_k| = |B_k| \le \frac{1}{2} C \, |(w_0 - w^*) \cdot v_k|.$$

By orthonormality of the basis,

$$\begin{aligned}|w_n - w^*|^2 &= \sum_k |A_k \rho_k^n + B_k \bar{\rho}_k^n|^2 \\ &\le \sum_k (|A_k| + |B_k|)^2 s^n \\ &\le C^2 s^n \sum_k |(w_0 - w^*) \cdot v_k|^2 \\ &= C^2 s^n |w_0 - w^*|^2.\end{aligned}$$

□

Note: Since the proof is dimension-independent, a version of the result should hold in Hilbert space.

References

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