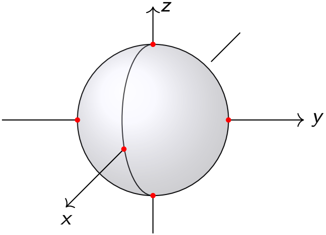
Note: The audio in this pdf won't play without additional files.

The Volume of the Ball in n Dimensions

Low Dimensional Balls

$$B^3 = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$$

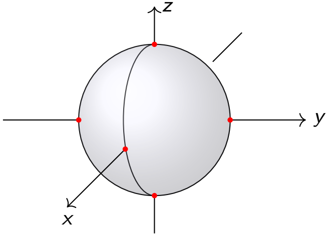
$$S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$



Low Dimensional Balls

$$B^3 = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$$

$$S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$$

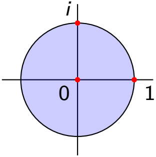


The volume of B^3 is $4\pi/3 = 4.19$ and the area of S^2 is 4π.

Low Dimensional Balls, Continued

$$B^2 = \{(x, y) : x^2 + y^2 \leq 1\}$$

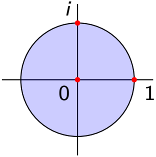
$$S^1 = \{(x, y) : x^2 + y^2 = 1\}$$



Low Dimensional Balls, Continued

$$B^2 = \{(x, y) : x^2 + y^2 \leq 1\}$$

$$S^1 = \{(x, y) : x^2 + y^2 = 1\}$$

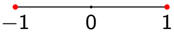


The area of B^2 is $\pi = 3.14$ and the length of S^1 is 2π.

Low Dimensional Balls, Continued

$$B^1 = \{x : x^2 \leq 1\}$$

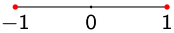
$$S^0 = \{x : x^2 = 1\}$$



Low Dimensional Balls, Continued

$$B^1 = \{x : x^2 \leq 1\}$$

$$S^0 = \{x : x^2 = 1\}$$



The length of $B^1 = [-1, 1]$ is 2 and the measure of S^0 is 2.

Low Dimensional Balls, Continued

$$B^0 = \{0\}$$

Low Dimensional Balls, Continued

$$B^0 = \{0\}$$

The measure of B^0 is 1.

The General Formula

In n dimensions, the ball and the sphere are

The General Formula

In n dimensions, the ball and the sphere are

$$B^n = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$$

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

The General Formula

In n dimensions, the ball and the sphere are

$$B^n = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$$

$$S^{n-1} = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

Throughout $|G|$ denotes the size, measure, volume, area, length, etc. of the geometric object G. Which it actually is will be clear from the context.

The General Formula

In n dimensions, the ball and the sphere are

$$B^n = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$$

$$S^{n-1} = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}.$$

Throughout $|G|$ denotes the size, measure, volume, area, length, etc. of the geometric object G. Which it actually is will be clear from the context. We will show

$$\boxed{|B^n| = \frac{\pi^{n/2}}{(n/2)!}}$$

$$\boxed{|B^n| = \frac{1}{n}|S^{n-1}|.}$$

History

$$\boxed{|B^n| = \frac{1}{n}\,|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}\,|S^{n-1}|}$$

is *Archimedes' formula*.

History

$$|B^n| = \frac{1}{n}|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}|S^{n-1}|$$

is *Archimedes' formula*. The idea behind this formula is *suspension*.

History

$$\boxed{|B^n| = \frac{1}{n}\,|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}\,|S^{n-1}|}$$

is *Archimedes' formula*. The idea behind this formula is *suspension*. The general formula

$$\boxed{|B^n| = \frac{\pi^{n/2}}{(n/2)!}}$$

goes back to the beginning of calculus.

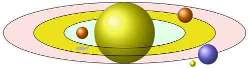
History

$$|B^n| = \frac{1}{n}|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}|S^{n-1}|$$

is *Archimedes' formula*. The idea behind this formula is *suspension*. The general formula

$$|B^n| = \frac{\pi^{n/2}}{(n/2)!}$$

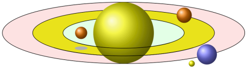
goes back to the beginning of calculus. At this point, Isaac would say ...



$$|B^n| = \frac{1}{n}\,|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}\,|S^{n-1}|$$

is *Archimedes' formula*. The idea behind this formula is *suspension*. The general formula

$$|B^n| = \frac{\pi^{n/2}}{(n/2)!}$$



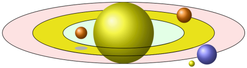
goes back to the beginning of calculus. At this point, Isaac would say . . . and Carl • would improve this by saying . . .

History

$$|B^n| = \frac{1}{n}\,|S^{n-1}| = \frac{\text{radius}}{\text{dimension}}\,|S^{n-1}|$$

is *Archimedes' formula*. The idea behind this formula is *suspension*. The general formula

$$|B^n| = \frac{\pi^{n/2}}{(n/2)!}$$



goes back to the beginning of calculus. At this point, Isaac would say ... and Carl • would improve this by saying ... then Archimedes • would respond ...

History, Continued

Archimedes lived in the city of Syracuse in Sicily. More than a century after the death of Archimedes, Cicero, the roman senator/lawyer/orator/etc, was governor of Sicily. In his memoirs, Cicero wrote

History, Continued

Archimedes lived in the city of Syracuse in Sicily. More than a century after the death of Archimedes, Cicero, the roman senator/lawyer/orator/etc, was governor of Sicily. In his memoirs, Cicero wrote

- *When I was questor in Sicily [in 75 BC, 137 years after the death of Archimedes] I managed to track down his grave. The Syracusians knew nothing about it, and indeed denied that any such thing existed. But there it was, completely surrounded and hidden by bushes of brambles and thorns.*
- *I remembered having heard of some simple lines of verse which had been inscribed on his tomb, referring to a sphere and cylinder modelled in stone on top of the grave. And so I took a good look round all the numerous tombs that stand beside the Agrigentine Gate. Finally I noted a little column just visible above the scrub: it was surmounted by a sphere and a cylinder.*
- *I immediately said to the Syracusans, some of whose leading citizens were with me at the time, that I believed this was the very object I had been looking for. Men were sent in with sickles to clear the site, and when a path to the monument had been opened we walked right up to it. And the verses were still visible, though approximately the second half of each line had been worn away.*
- *So one of the most famous cities in the Greek world, and in former days a great centre of learning as well, would have remained in total ignorance of the tomb of the most brilliant citizen it had ever produced, had a man from Arpinum not come and pointed it out!*

History, Continued

- What does $x!$ mean?

- What does $x!$ mean? As Ada puts it ...

Factorials and π $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- What does $x!$ mean? As Ada puts it ...
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.

Factorials and π $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- What does $x!$ mean? As Ada puts it ...
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.
- By the general formula for $n = 0$, $0! = 1$ (hence $1! = 1 \cdot 0! = 1$).

- What does $x!$ mean? As Ada puts it ...
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.
- By the general formula for $n = 0$, $0! = 1$ (hence $1! = 1 \cdot 0! = 1$).
- What does π mean? By the general formula for $n = 2$, π is the area of the disk $D = B^2$.

Factorials and π $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- What does $x!$ mean? As Ada puts it . . .
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.
- By the general formula for $n = 0$, $0! = 1$ (hence $1! = 1 \cdot 0! = 1$).
- What does π mean? By the general formula for $n = 2$, π is the area of the disk $D = B^2$.
- What does $(1/2)!$ mean? By the general formula for $n = 1$, $(1/2)! = \sqrt{\pi}/2$ (hence $(-1/2)! = \sqrt{\pi}$).

Factorials and π $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- What does $x!$ mean? As Ada puts it ...
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.
- By the general formula for $n = 0$, $0! = 1$ (hence $1! = 1 \cdot 0! = 1$).
- What does π mean? By the general formula for $n = 2$, π is the area of the disk $D = B^2$.
- What does $(1/2)!$ mean? By the general formula for $n = 1$, $(1/2)! = \sqrt{\pi}/2$ (hence $(-1/2)! = \sqrt{\pi}$).
- Thus we are *defining* $0!$, π, and $(1/2)!$ by the general formula for $n = 0$, $n = 2$, and $n = 1$.

- What does $x!$ mean? As Ada puts it ...
- The factorial $x!$ is a function satisfying $x! = x \cdot (x-1)!$.
- By the general formula for $n = 0$, $0! = 1$ (hence $1! = 1 \cdot 0! = 1$).
- What does π mean? By the general formula for $n = 2$, π is the area of the disk $D = B^2$.
- What does $(1/2)!$ mean? By the general formula for $n = 1$, $(1/2)! = \sqrt{\pi}/2$ (hence $(-1/2)! = \sqrt{\pi}$).
- Thus we are *defining* $0!$, π, and $(1/2)!$ by the general formula for $n = 0$, $n = 2$, and $n = 1$.
- Of course, Leonhard • would say ...

Factorials and π, Continued

With these definitions, we have

$$1! = 1, \qquad 2! = 2, \qquad 3! = 6, \ldots,$$
$$(3/2)! = 3\sqrt{\pi}/4, \qquad (5/2)! = 15\sqrt{\pi}/8, \qquad (7/2)! = 105\sqrt{\pi}/16, \ldots,$$

and the general formula is meaningful for all $n \geq 0$.

Factorials and π, Continued $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

With these definitions, we have

$$1! = 1, \qquad 2! = 2, \qquad 3! = 6, \dots,$$
$$(3/2)! = 3\sqrt{\pi}/4, \qquad (5/2)! = 15\sqrt{\pi}/8, \qquad (7/2)! = 105\sqrt{\pi}/16, \dots,$$

and the general formula is meaningful for all $n \geq 0$. The general formula then implies $|B^3| = 4\pi/3$,

$$|B^4| = \pi^2/2 = 4.93, |B^5| = 8\pi^2/15 = 5.26, |B^6| = \pi^3/6 = 5.16.$$

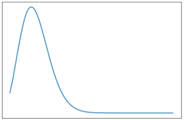
Factorials and π, Continued

With these definitions, we have

$$1! = 1, \qquad 2! = 2, \qquad 3! = 6, \dots,$$
$$(3/2)! = 3\sqrt{\pi}/4, \qquad (5/2)! = 15\sqrt{\pi}/8, \qquad (7/2)! = 105\sqrt{\pi}/16, \dots,$$

and the general formula is meaningful for all $n \geq 0$. The general formula then implies $|B^3| = 4\pi/3$,

$$|B^4| = \pi^2/2 = 4.93, |B^5| = 8\pi^2/15 = 5.26, |B^6| = \pi^3/6 = 5.16.$$



Plan of Attack $|B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.

Plan of Attack $\quad |B^n| = \frac{\pi^{n/2}}{(n/2)!}$

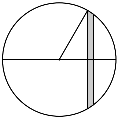
- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

The first part follows from slicing against the last coordinate x in $\mathbf{R}^{2n+1}$

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

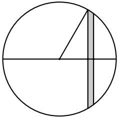
The first part follows from slicing against the last coordinate x in $\mathbf{R}^{2n+1}$



Plan of Attack $|B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

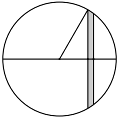
The first part follows from slicing against the last coordinate x in $\mathbf{R}^{2n+1}$



$$\frac{|B^{2n+1}|}{|B^{2n}|} = \int_{-1}^{1} (1-x^2)^n \, dx$$

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

The first part follows from slicing against the last coordinate x in $\mathbf{R}^{2n+1}$

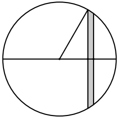


$$\frac{|B^{2n+1}|}{|B^{2n}|} = \int_{-1}^{1} (1-x^2)^n \, dx = \sqrt{\pi} \frac{n!}{(n+1/2)!}.$$

Plan of Attack $|B^n| = \frac{\pi^{n/2}}{(n/2)!}$

- Show the general formula for $2n+1$ is a consequence of the general formula for $2n$.
- Derive the general formula for $2n$.

The first part follows from slicing against the last coordinate x in $\mathbf{R}^{2n+1}$



$$\frac{|B^{2n+1}|}{|B^{2n}|} = \int_{-1}^{1} (1-x^2)^n \, dx = \sqrt{\pi}\frac{n!}{(n+1/2)!}.$$

So now we have to derive the general formula for $2n$ with $n \geq 2$.

Even Dimensional Balls

Look at the even case

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$

Even Dimensional Balls

Look at the even case

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$

Every formula goes with a picture. Every picture goes with a formula. What's the picture here?

Even Dimensional Balls

Look at the even case

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$

Every formula goes with a picture. Every picture goes with a formula. What's the picture here?
To get the correct picture, we need to look at the ball as a geometrical object in complex space, not real space.

Look at the even case

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$

Every formula goes with a picture. Every picture goes with a formula. What's the picture here?

To get the correct picture, we need to look at the ball as a geometrical object in complex space, not real space. As Maryam

- says ...

Even Dimensional Balls

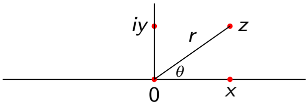
Look at the even case

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$

Every formula goes with a picture. Every picture goes with a formula. What's the picture here?

To get the correct picture, we need to look at the ball as a geometrical object in complex space, not real space. As Maryam • says ... For $z = x + iy$, introduce polar coordinates,

$$|z|^2 = r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = re^{i\theta}$$



r is the *radius* and θ is the *angle*.

Complex Coordinates

To derive the even case,

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

Complex Coordinates

To derive the even case,

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

we identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ via

$$(x_1, y_1, \dots, x_n, y_n) \quad \leftrightarrow \quad (x_1 + iy_1, \dots, x_n + iy_n) = (z_1, \dots, z_n).$$

Complex Coordinates

To derive the even case,

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

we identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ via

$$(x_1, y_1, \ldots, x_n, y_n) \quad \leftrightarrow \quad (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n).$$

It's difficult to graph objects in $\mathbf{C}^n$ for $n > 1$.
But the map

$$(z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_n)$$

projects $\mathbf{C}^n$ into $\mathbf{R}^n$, allowing us a partial visualization, called *radial space*.

Complex Coordinates

To derive the even case,

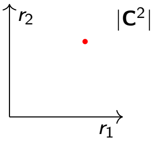
$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

we identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ via

$$(x_1, y_1, \ldots, x_n, y_n) \quad \leftrightarrow \quad (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n).$$

It's difficult to graph objects in $\mathbf{C}^n$ for $n > 1$. But the map

$$(z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_n)$$



projects $\mathbf{C}^n$ into $\mathbf{R}^n$, allowing us a partial visualization, called *radial space*.

Complex Coordinates

To derive the even case,

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

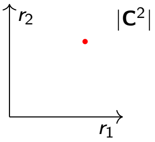
we identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ via

$$(x_1, y_1, \ldots, x_n, y_n) \quad \leftrightarrow \quad (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n).$$

It's difficult to graph objects in $\mathbf{C}^n$ for $n > 1$. But the map

$$(z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_n)$$

projects $\mathbf{C}^n$ into $\mathbf{R}^n$, allowing us a partial visualization, called *radial space*. The inverse image of a point in radial space is an n-torus.



Complex Coordinates

To derive the even case,

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!},}$$

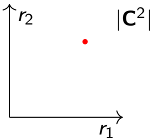
we identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ via

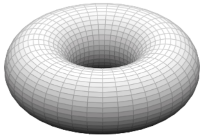
$$(x_1, y_1, \ldots, x_n, y_n) \quad \leftrightarrow \quad (x_1 + iy_1, \ldots, x_n + iy_n) = (z_1, \ldots, z_n).$$

It's difficult to graph objects in $\mathbf{C}^n$ for $n > 1$. But the map

$$(z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_n)$$

projects $\mathbf{C}^n$ into $\mathbf{R}^n$, allowing us a partial visualization, called *radial space*. The inverse image of a point in radial space is an n-torus.





The Polydisk

In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.

The Polydisk

In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \ldots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$

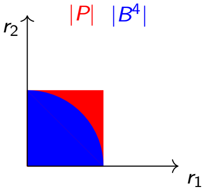
The Polydisk

In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \ldots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$



The Polydisk

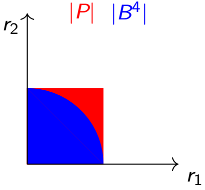
In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \ldots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$

For $n = 1$, $B^{2n} = P = D$. For all $n > 1$, B^{2n} and P are *topologically* equivalent.



The Polydisk

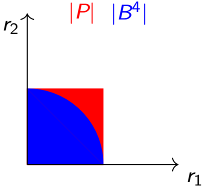
In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \dots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \dots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \dots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$

For $n = 1$, $B^{2n} = P = D$. For all $n > 1$, B^{2n} and P are *topologically* equivalent. For all $n > 1$, B^{2n} and P are *smoothly* equivalent.



The Polydisk

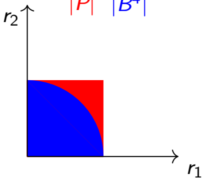
In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \ldots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$

For $n = 1$, $B^{2n} = P = D$. For all $n > 1$, B^{2n} and P are *topologically* equivalent. For all $n > 1$, B^{2n} and P are *smoothly* equivalent. But as Henri • points out ...



The Polydisk

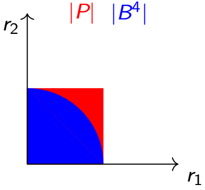
In $\mathbf{C}^n$,

$$B^{2n} = \left\{(z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_n|^2 \leq 1\right\}.$$

Let $D = B^2 = \{z : |z|^2 \leq 1\}$ be the disk in $\mathbf{C}$.
Let $P = D^n$ be the *polydisk* in $\mathbf{C}^n$, the *n-th power* of D,

$$P = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 \leq 1, \ldots, |z_n|^2 \leq 1\} = D \times \cdots \times D.$$

For $n = 1$, $B^{2n} = P = D$. For all $n > 1$, B^{2n} and P are *topologically* equivalent. For all $n > 1$, B^{2n} and P are *smoothly* equivalent. But as Henri • points out ... Then Olga • responds ...



Multiplicativity of Volume

Let A_1 and A_2 be regions in euclidean space and let $A_1 \times A_2$ be their cartesian product.

Multiplicativity of Volume

Let A_1 and A_2 be regions in euclidean space and let $A_1 \times A_2$ be their cartesian product. Then the volume of the product equals the product of the volumes,

$$|A_1 \times A_2| = |A_1| \cdot |A_2|,$$

Multiplicativity of Volume

Let A_1 and A_2 be regions in euclidean space and let $A_1 \times A_2$ be their cartesian product. Then the volume of the product equals the product of the volumes,

$$|A_1 \times A_2| = |A_1| \cdot |A_2|,$$

so we have $|P| = |D^n| = |D \times \cdots \times D| = \pi^n$.

Multiplicativity of Volume

Let A_1 and A_2 be regions in euclidean space and let $A_1 \times A_2$ be their cartesian product. Then the volume of the product equals the product of the volumes,

$$|A_1 \times A_2| = |A_1| \cdot |A_2|,$$

so we have $|P| = |D^n| = |D \times \cdots \times D| = \pi^n$.
But then Sofya • would say ...

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P.

Permutations $P = D \times \cdots \times D$

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P.

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes,

Permutations $P = D \times \cdots \times D$

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \dots, z_n) \to (z_{g1}, \dots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G,

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D,

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

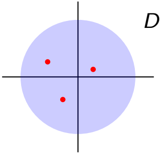
$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \ldots, z_n$ in D,

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \dots, z_n) \to (z_{g1}, \dots, z_{gn}).$$

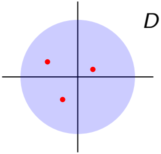
This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \dots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \dots, z_n$ in D,



Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \ldots, z_n$ in D, the configuration space of n bosons $z_1, \ldots, z_n$ in D.

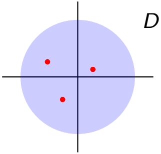


Permutations $\qquad P = D \times \cdots \times D$

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \ldots, z_n$ in D, the configuration space of n bosons $z_1, \ldots, z_n$ in D.

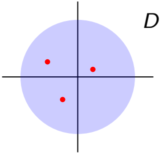


$P/G = D \vee \cdots \vee D$ is the symmetric n-th power of D.

Permutations $P = D \times \cdots \times D$

Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$



This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \ldots, z_n$ in D, the configuration space of n bosons $z_1, \ldots, z_n$ in D.

$P/G = D \vee \cdots \vee D$ is the symmetric n-th power of D.

$S^1 \vee S^1$

Permutations $P = D \times \cdots \times D$

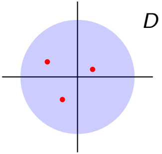
Let G be the group of *permutations* g on n letters. Then $|G| = n!$, and there is an *action* of G on P: Each g in G *permutes the coordinates,*

$$(z_1, \ldots, z_n) \to (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on P. Let P/G be the collection of equivalence classes, the orbit space under G, the collection of unordered n-tuples $< z_1, \ldots, z_n >$, with each coordinate in D, the configuration space of n points $z_1, \ldots, z_n$ in D, the configuration space of n bosons $z_1, \ldots, z_n$ in D.

$P/G = D \vee \cdots \vee D$ is the symmetric n-th power of D.

At this point, Emmy • interjects . . .



$S^1 \vee S^1$

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \ldots, z_n) \in P : 0 \leq |z_1|^2 \leq |z_2|^2 \leq \cdots \leq |z_n|^2 \leq 1\},$$

Permutations, Continued $|B^{2n}| = \frac{\pi^n}{n!}$

In fact, P/G is in one-to-one correspondence with

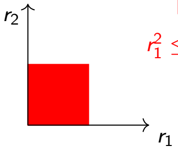
$$P_1 = \{(z_1, \dots, z_n) \in P : 0 \le |z_1|^2 \le |z_2|^2 \le \cdots \le |z_n|^2 \le 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.

In fact, P/G is in one-to-one correspondence with

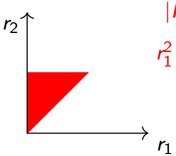
$$P_1 = \{(z_1, \ldots, z_n) \in P : 0 \leq |z_1|^2 \leq |z_2|^2 \leq \cdots \leq |z_n|^2 \leq 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.



$|P|$

$r_1^2 \leq 1, r_2^2 \leq 1$



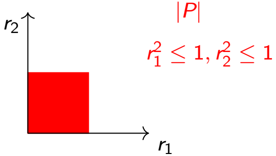
$|P_1|$

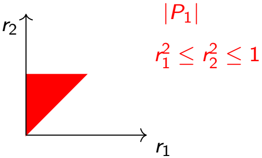
$r_1^2 \leq r_2^2 \leq 1$

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \dots, z_n) \in P : 0 \le |z_1|^2 \le |z_2|^2 \le \cdots \le |z_n|^2 \le 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.





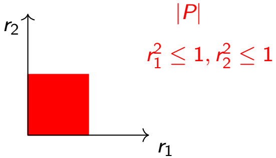
In other words, G *tessellates* P into cells $P_g = g(P_1)$, $g \in G$.

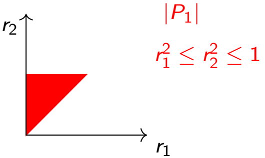
Permutations, Continued $\quad |B^{2n}| = \frac{\pi^n}{n!}$

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \ldots, z_n) \in P : 0 \leq |z_1|^2 \leq |z_2|^2 \leq \cdots \leq |z_n|^2 \leq 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.



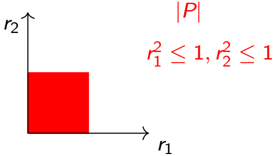


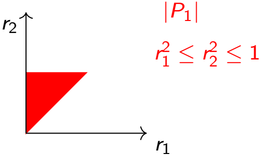
In other words, G *tessellates* P into cells $P_g = g(P_1)$, $g \in G$. Since the action of G on P is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \pi^n/n!$.

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \ldots, z_n) \in P : 0 \le |z_1|^2 \le |z_2|^2 \le \cdots \le |z_n|^2 \le 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.





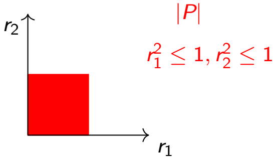
In other words, G *tessellates* P into cells $P_g = g(P_1)$, $g \in G$. Since the action of G on P is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \pi^n/n!$. Evariste • would say . . .

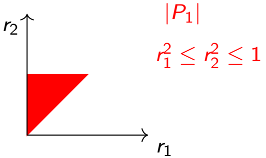
Permutations, Continued $\qquad |B^{2n}| = \frac{\pi^n}{n!}$

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \dots, z_n) \in P : 0 \le |z_1|^2 \le |z_2|^2 \le \cdots \le |z_n|^2 \le 1\},$$

and P_1 is a *fundamental domain* for the action of G on P.



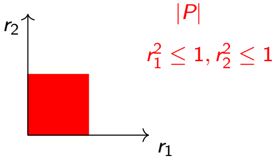


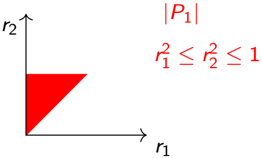
In other words, G *tessellates* P into cells $P_g = g(P_1)$, $g \in G$. Since the action of G on P is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \pi^n/n!$. Evariste • would say ... Thus it remains to find a *volume-preserving* bijection between P_1 and B^{2n}.

In fact, P/G is in one-to-one correspondence with

$$P_1 = \{(z_1, \dots, z_n) \in P : 0 \leq |z_1|^2 \leq |z_2|^2 \leq \cdots \leq |z_n|^2 \leq 1\},$$

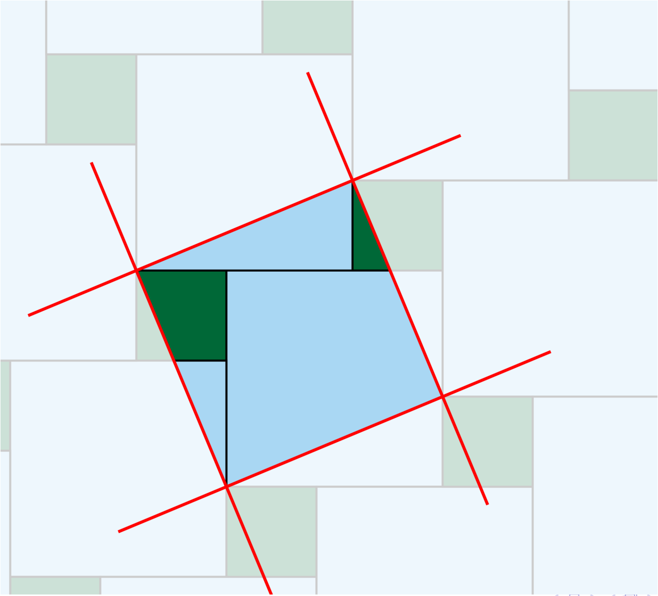
and P_1 is a *fundamental domain* for the action of G on P.





In other words, G *tessellates* P into cells $P_g = g(P_1)$, $g \in G$. Since the action of G on P is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \pi^n/n!$. Evariste • would say ... Thus it remains to find a *volume-preserving* bijection between P_1 and B^{2n}. But, before we do that, Pythagoras • complains ...

The Pythagorean Tessellation



Volume Preserving Maps

Let $z' = (z'_1, \ldots, z'_n)$ denote a point in B^{2n}, and $z = (z_1, \ldots, z_n)$ a point in P_1. We seek a bijective volume-preserving map between z in P_1 and z' in B^{2n}.

Volume Preserving Maps

Let $z' = (z'_1, \ldots, z'_n)$ denote a point in B^{2n}, and $z = (z_1, \ldots, z_n)$ a point in P_1. We seek a bijective volume-preserving map between z in P_1 and z' in B^{2n}.

Let $r_1, \ldots, r_n$ and $\theta_1, \ldots, \theta_n$ be the radii and angles of $z_1, \ldots, z_n$.

Volume Preserving Maps

Let $z' = (z'_1, \ldots, z'_n)$ denote a point in B^{2n}, and $z = (z_1, \ldots, z_n)$ a point in P_1. We seek a bijective volume-preserving map between z in P_1 and z' in B^{2n}.
Let $r_1, \ldots, r_n$ and $\theta_1, \ldots, \theta_n$ be the radii and angles of $z_1, \ldots, z_n$. The map $z \leftrightarrow z'$ we seek is *defined* by

$$\theta_1 = \theta'_1, \theta_2 = \theta'_2, \ldots, \theta_n = \theta'_n$$

and

$$\begin{aligned}
{r_1}^2 &= {r'_1}^2 \\
{r_2}^2 &= {r'_1}^2 + {r'_2}^2 \\
&\ldots \\
{r_n}^2 &= {r'_1}^2 + {r'_2}^2 + {r'_3}^2 + \cdots + {r'_n}^2.
\end{aligned}$$

Volume Preserving Maps

Let $z' = (z'_1, \ldots, z'_n)$ denote a point in B^{2n}, and $z = (z_1, \ldots, z_n)$ a point in P_1. We seek a bijective volume-preserving map between z in P_1 and z' in B^{2n}.
Let $r_1, \ldots, r_n$ and $\theta_1, \ldots, \theta_n$ be the radii and angles of $z_1, \ldots, z_n$. The map $z \leftrightarrow z'$ we seek is *defined* by

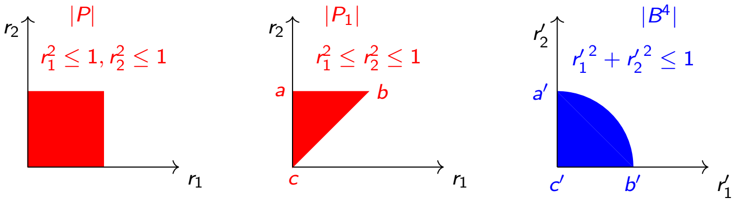
$$\theta_1 = \theta'_1, \theta_2 = \theta'_2, \ldots, \theta_n = \theta'_n$$

and

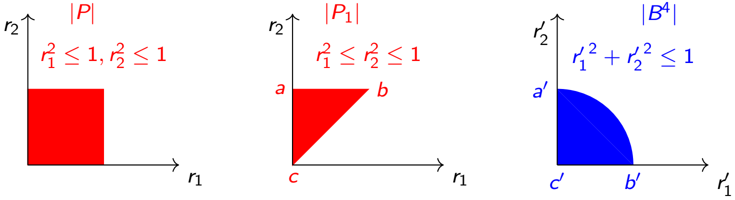
$$\begin{aligned} {r_1}^2 &= {r'_1}^2 \\ {r_2}^2 &= {r'_1}^2 + {r'_2}^2 \\ &\cdots \\ {r_n}^2 &= {r'_1}^2 + {r'_2}^2 + {r'_3}^2 + \cdots + {r'_n}^2. \end{aligned}$$

This map is certainly a bijection between z in P_1 and z' in B^{2n}.

Volume Preserving Maps, Continued



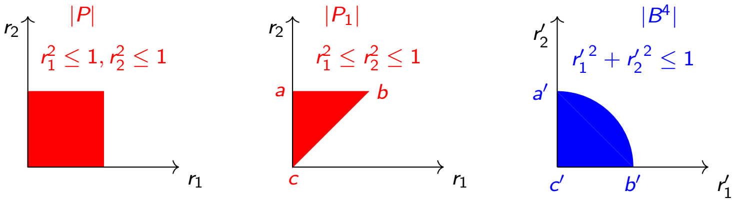
Volume Preserving Maps, Continued



Now a map $z \mapsto z'$ in $\mathbf{C}$ is area-preserving if the area elements agree

$$dxdy = r\,drd\theta = r'\,dr'\,d\theta' = dx'\,dy'.$$

Volume Preserving Maps, Continued

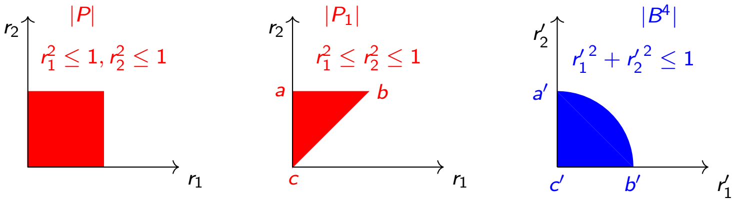


Now a map $z \mapsto z'$ in $\mathbf{C}$ is area-preserving if the area elements agree

$$dxdy = r\,drd\theta = r'\,dr'\,d\theta' = dx'\,dy'.$$

In particular, if $d\theta' = d\theta$ and $r'\,dr' = rdr$, the map is area preserving.

Volume Preserving Maps, Continued



Now a map $z \mapsto z'$ in $\mathbf{C}$ is area-preserving if the area elements agree

$$dxdy = r\,drd\theta = r'\,dr'\,d\theta' = dx'\,dy'.$$

In particular, if $d\theta' = d\theta$ and $r'\,dr' = rdr$, the map is area preserving. Similarly, a map $z \mapsto z'$ in $\mathbf{C}^n$ is volume-preserving if

$$d\theta_1' d\theta_2' \dots d\theta_n' = d\theta_1 d\theta_2 \dots d\theta_n,$$

$$r_1'\,dr_1'\,r_2'\,dr_2'\,\dots\,r_n'\,dr_n' = r_1\,dr_1\,r_2\,dr_2\,\dots\,r_n\,dr_n.$$

Volume Preserving Maps, Continued

But our map preserves coordinate angles and satisfies

$$
\begin{aligned}
{r_1}^2 &= {r_1'}^2 \\
{r_2}^2 &= {r_1'}^2 + {r_2'}^2 \\
{r_3}^2 &= {r_1'}^2 + {r_2'}^2 + {r_3'}^2, \\
&\dots \\
{r_n}^2 &= {r_1'}^2 + {r_2'}^2 + {r_3'}^2 + \cdots + {r_n'}^2.
\end{aligned}
$$

Volume Preserving Maps, Continued

But our map preserves coordinate angles and satisfies

$$\begin{aligned} {r_1}^2 &= {r_1'}^2 \\ {r_2}^2 &= {r_1'}^2 + {r_2'}^2 \\ {r_3}^2 &= {r_1'}^2 + {r_2'}^2 + {r_3'}^2, \\ &\dots \\ {r_n}^2 &= {r_1'}^2 + {r_2'}^2 + {r_3'}^2 + \dots + {r_n'}^2. \end{aligned}$$

Differentiating yields

$$\begin{aligned} r_1 dr_1 &= r_1' dr_1' \\ r_2 dr_2 &= r_1' dr_1' + r_2' dr_2' \\ r_3 dr_3 &= r_1' dr_1' + r_2' dr_2' + r_3' dr_3', \\ &\dots \\ r_n dr_n &= r_1' dr_1' + r_2' dr_2' + r_3' dr_3' + \dots + r_n' dr_n'. \end{aligned}$$

Volume Preserving Maps, Continued

Since this is a sequence of (infinitesimal) shears,

$$r'_1\,dr'_1\;r'_2\,dr'_2\;\dots\;r'_n\,dr'_n = r_1\,dr_1\;r_2\,dr_2\;\dots\;r_n\,dr_n,$$

Volume Preserving Maps, Continued

Since this is a sequence of (infinitesimal) shears,

$$r'_1\,dr'_1\;r'_2\,dr'_2\;\ldots\;r'_n\,dr'_n = r_1\,dr_1\;r_2\,dr_2\;\ldots\;r_n\,dr_n,$$

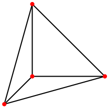
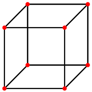
and we're done.

Volume Preserving Maps, Continued

Since this is a sequence of (infinitesimal) shears,

$$r_1' \, dr_1' \; r_2' \, dr_2' \; \dots \; r_n' \, dr_n' = r_1 \, dr_1 \; r_2 \, dr_2 \; \dots \; r_n \, dr_n,$$

and we're done. Setting $t_1 = r_1^2/2, \dots, t_n = r_n^2/2$, all of the above is the same as saying

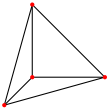
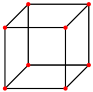
B^{2n} =  $\times$  $(n = 3)$

Volume Preserving Maps, Continued

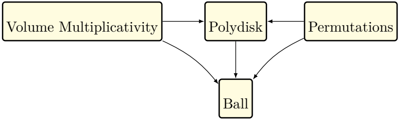
Since this is a sequence of (infinitesimal) shears,

$$r_1' \, dr_1' \, r_2' \, dr_2' \, \dots \, r_n' \, dr_n' = r_1 \, dr_1 \, r_2 \, dr_2 \, \dots \, r_n \, dr_n,$$

and we're done. Setting $t_1 = r_1^2/2, \dots, t_n = r_n^2/2$, all of the above is the same as saying

B^{2n} $=$  $\times$  $(n = 3)$

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!}}$$



Odd Dimensional Spheres

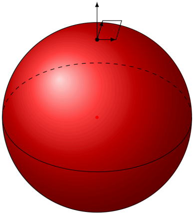
The sphere S^{2n+1} consists of all points $(z_1, \dots, z_n, z_{n+1}) = (z, z_{n+1})$ in $\mathbf{C}^{n+1}$ satisfying

$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$

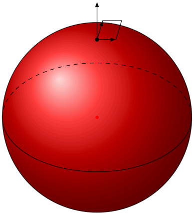
Odd Dimensional Spheres

The sphere S^{2n+1} consists of all points $(z_1, \dots, z_n, z_{n+1}) = (z, z_{n+1})$ in $\mathbf{C}^{n+1}$ satisfying

$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$



Odd Dimensional Spheres



The sphere S^{2n+1} consists of all points $(z_1, \ldots, z_n, z_{n+1}) = (z, z_{n+1})$ in $\mathbf{C}^{n+1}$ satisfying

$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$

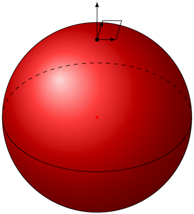
Using $r_1 dr_1 + \ldots r_n dr_n + r_{n+1} dr_{n+1} = 0$, the map

$$(z, \theta) \to \left(z, \theta, \sqrt{1 - (r_1^2 + \cdots + r_n^2)} \right)$$

is a volume-preserving bijection

$$B^{2n} \times S^1 \to S^{2n+1}.$$

Odd Dimensional Spheres



The sphere S^{2n+1} consists of all points $(z_1, \dots, z_n, z_{n+1}) = (z, z_{n+1})$ in $\mathbf{C}^{n+1}$ satisfying

$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$

Using $r_1 dr_1 + \dots r_n dr_n + r_{n+1} dr_{n+1} = 0$, the map

$$(z, \theta) \to \left(z, \theta, \sqrt{1 - (r_1^2 + \cdots + r_n^2)} \right)$$

is a volume-preserving bijection

$$B^{2n} \times S^1 \to S^{2n+1}.$$

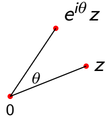
Hence $|S^{2n+1}| = |S^1| \, |B^{2n}|$.

Complex Projective Space

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \ldots, z_n, z_{n+1}) \to (z_1, \ldots, z_n, e^{i\theta} z_{n+1})$$

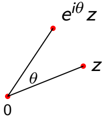
has orbit space B^{2n}.



Complex Projective Space

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \ldots, z_n, z_{n+1}) \to (z_1, \ldots, z_n, e^{i\theta} z_{n+1})$$



has orbit space B^{2n}. Now the circle action on S^{2n+1}

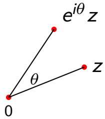
$$(z_1, \ldots, z_n, z_{n+1}) \to (e^{i\theta} z_1, \ldots, e^{i\theta} z_n, e^{i\theta} z_{n+1})$$

has orbit space $\mathbf{C}P^n$

Complex Projective Space

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \dots, z_n, z_{n+1}) \to (z_1, \dots, z_n, e^{i\theta} z_{n+1})$$



has orbit space B^{2n}. Now the circle action on S^{2n+1}

$$(z_1, \dots, z_n, z_{n+1}) \to (e^{i\theta} z_1, \dots, e^{i\theta} z_n, e^{i\theta} z_{n+1})$$

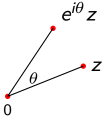
has orbit space $\mathbf{C}P^n$, so we have projections

$$\begin{tikzcd} & S^{2n+1} \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\ B^{2n} & & \mathbf{C}P^n \end{tikzcd}$$

Complex Projective Space

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \dots, z_n, z_{n+1}) \to (z_1, \dots, z_n, e^{i\theta} z_{n+1})$$



has orbit space B^{2n}. Now the circle action on S^{2n+1}

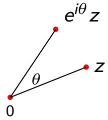
$$(z_1, \dots, z_n, z_{n+1}) \to (e^{i\theta} z_1, \dots, e^{i\theta} z_n, e^{i\theta} z_{n+1})$$

has orbit space $\mathbf{C}P^n$, so we have projections

$$\begin{tikzcd}
 & S^{2n+1} \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\
B^{2n} & & \mathbf{C}P^n
\end{tikzcd}$$

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \dots, z_n, z_{n+1}) \to (z_1, \dots, z_n, e^{i\theta} z_{n+1})$$



has orbit space B^{2n}. Now the circle action on S^{2n+1}

$$(z_1, \dots, z_n, z_{n+1}) \to (e^{i\theta} z_1, \dots, e^{i\theta} z_n, e^{i\theta} z_{n+1})$$

has orbit space $\mathbf{C}P^n$, so we have projections

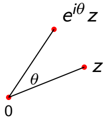
$$\begin{tikzcd} & S^{2n+1} \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\ B^{2n} & & \mathbf{C}P^n \end{tikzcd}$$

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!} = |\mathbf{C}P^n|}$$

Complex Projective Space

Rephrasing the previous slide, the circle action on S^{2n+1}

$$(z_1, \ldots, z_n, z_{n+1}) \to (z_1, \ldots, z_n, e^{i\theta} z_{n+1})$$



has orbit space B^{2n}. Now the circle action on S^{2n+1}

$$(z_1, \ldots, z_n, z_{n+1}) \to (e^{i\theta} z_1, \ldots, e^{i\theta} z_n, e^{i\theta} z_{n+1})$$

has orbit space $\mathbf{C}P^n$, so we have projections

$$\begin{tikzcd} & S^{2n+1} \arrow[dl, "S^1"'] \arrow[dr, "S^1"] & \\ B^{2n} & & \mathbf{C}P^n \end{tikzcd}$$

$$\boxed{|B^{2n}| = \frac{\pi^n}{n!} = |\mathbf{C}P^n|}$$